ON THE GRADED CENTER OF THE STABLE CATEGORY OF A FINITE p-GROUP

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ABSTRACT. We show that for any finite p-group P of rank at least 2 and any algebraically closed field k of characteristic p the graded center $Z^*(\underline{\mathrm{mod}}(kP))$ of the stable module category of finite-dimensional kP-modules has infinite dimension in each odd degree, and if $p = 2$ also in each even degree. In particular, this provides examples of symmetric algebras A for which $Z^0(\underline{\mathrm{mod}}(A))$ is not finite-dimensional, answering a question raised in [10].

1. INTRODUCTION

The graded center of a k-linear triangulated category $(\mathcal{C}; \Sigma)$ over a commutative ring k is the graded k-module $Z^*(\mathcal{C}) = Z^*(\mathcal{C}; \Sigma)$ which in degree $n \in \mathbb{Z}$ consists of all k-linear natural transformations $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ satisfying $\Sigma\varphi = (-1)^n \varphi\Sigma$; this becomes a graded commutative k-algebra with multiplication essentially induced by composition in $\mathcal{C}$. We refer to [10] for more details. By [5], the stable category $\underline{\mathrm{mod}}(A)$ of finitely generated modules over a finite-dimensional self-injective algebra A is a triangulated category with shift functor the inverse of the Heller operator. Its graded center has been calculated for Brauer tree algebras [7] and in particular also uniserial algebras [8]. These calculations suggest that Tate cohomology rings of blocks and the graded centers of their stable module categories are closely related. Since the Tate cohomology of a block is an invariant of the fusion system of the block, a good understanding of graded centers might shed some light on the question to what extent the fusion system of a block is determined by its stable module category. These calculations also suggest that in order to determine the graded center of the stable module category of a block one will need to do this first for a defect group algebra of the block, hence for finite p-group algebras. This is what motivates the present paper. The following result shows that $Z^0(\underline{\mathrm{mod}}(A))$ need not be finite-dimensional, answering a question raised in [10].

Theorem 1.1. *Let P be a finite 2-group of rank at least 2 and k an algebraically closed field of characteristic 2. Evaluation at the trivial kP-module induces a surjective homomorphism of graded k-algebras $Z^*(\underline{\mathrm{mod}}(kP)) \longrightarrow \hat{H}^*(P; k)$ whose kernel $\mathcal{I}$ is a nilpotent homogeneous ideal which is infinite-dimensional in each degree; in particular, $Z^0(\underline{\mathrm{mod}}(kP))$ has infinite dimension.*

For odd p we have a slightly weaker statement:

Theorem 1.2. *Let p be an odd prime, P a finite p-group of rank at least 2 and k an algebraically closed field of characteristic p. Evaluation at the trivial kP-module induces a surjective homomorphism of graded k-algebras $Z^*(\underline{\mathrm{mod}}(kP)) \longrightarrow \hat{H}^*(P; k)$ whose kernel $\mathcal{I}$ is a nilpotent homogeneous ideal which is infinite-dimensional in each odd degree.*

It is easy to see that the canonical map $Z^*(\underline{\mathrm{mod}}(kP)) \to \hat{H}^*(P; k)$ is surjective with nilpotent kernel $\mathcal{I}$ (see Lemma 2.7 below). The point of the above theorem is that this kernel tends to have infinite dimension in each degree (if $p = 2$) and at least in each odd degree if $p > 2$. Note though that for p odd there is no known example with $Z^0(\underline{\mathrm{mod}}(kP))$ having infinite dimension. The proof shows more precisely that these dimensions have as lower bound the cardinality of the field k. Technically, the proofs of the above theorems are based on the fact that for A a symmetric algebras, an almost split sequence ending in an indecomposable non projective A-module U determines an *almost vanishing morphism* $\zeta_U : U \to \Omega(U)$ which in turn provides elements of degree -1 in the graded center of the stable category; see e.g. [10, Proposition 1.4] or section 2 below. Using modules with appropriate periods, these can then be "shifted" to all other degrees if the underlying characteristic is 2 and all other odd degrees if the characteristic is odd. The elements of $Z^*(\underline{\mathrm{mod}}(A))$ obtained in this way will be called *almost vanishing*; see 2.5 for details. For Klein four groups, almost split sequences turn out to be the only way to obtain elements in the graded center of its stable module category beyond Tate cohomology. This can be seen using the classification of indecomposable modules over Klein four groups and leads to a slightly more precise statement.

Theorem 1.3. *Let P be a Klein four group and let k be an algebraically closed field of characteristic 2. Then the evaluation at the trivial kP-module induces a surjective homomorphism of graded k-algebras $Z^*(\underline{\mathrm{mod}}(kP)) \longrightarrow \hat{H}^*(P; k)$ whose kernel $\mathcal{I}$ is a homogeneous ideal which is infinite-dimensional in each degree. Moreover, we have $\mathcal{I}^2 = \{0\}$ and all elements in $\mathcal{I}$ are almost vanishing.*

This raises the question for which finite p-groups P is the graded center $Z^*(\underline{\mathrm{mod}}(kP))$ generated by Tate cohomology and almost vanishing elements. Another interesting question underlying some of the techical details below is the following. Given a periodic module U of period n of a symmetric algebra A over a field k, any isomorphism $\alpha : U \cong \Omega^n(U)$ induces an algebra automorphism of the stable endomorphism algebra $\underline{\mathrm{End}}_A(U)$ sending φ to $\alpha^{-1} \circ \Omega(\varphi) \circ \alpha$. When is this an inner automorphism? Equivalently, when can α be chosen in such a way that this automorphism is the identity? D. J. Benson [3] observed that the answer is positive if α is induced by an element in Tate cohomology (or the Tate analogue of Hochschild cohomology) because then α is the evaluation at U of a natural transformation from the identity functor on $\underline{\mathrm{mod}}(A)$ to the functor Ω^n.

2. Background material

Let A be a finite-dimensional algebra over a field k. The stable module category $\underline{\mathrm{mod}}(A)$ has as objects the finitely generated A-modules and as morphisms, for any two finitely generated A-modules U, V, the k-vector space $\underline{\mathrm{Hom}}_A(U, V) = \mathrm{Hom}_A(U, V)/\mathrm{Hom}_A^{pr}(U, V)$, where $\mathrm{Hom}_A^{pr}(U, V)$ is the k-vector space of A-homomorphisms from U to V which factor through a projective A-module, with composition of morphisms induced by the usual composition of A-homomorphisms in the category of finitely generated A-modules $\mathrm{mod}(A)$. The Heller operator Ω_A sends an A-module U to the kernel of a projective cover $P_U \to U$ of U; this induces a functor, still denoted Ω_A, on $\underline{\mathrm{mod}}(A)$, which is unique up to unique isomorphism of functors. If A is self-injective then Ω_A is an equivalence, and $\underline{\mathrm{mod}}(A)$ becomes a triangulated category with shift functor $\Sigma_A = \Omega_A^{-1}$, sending an A-module to a cokernel of an injective hull of U, with exact triangles induced by short exact sequences in $\mathrm{mod}(A)$. See e.g. [5] for details. If A is symmetric - that is, A is isomorphic, as A-A-bimodule to its k-dual $\mathrm{Hom}_k(A, k)$ - then A is

in particular self-injective. The literature on finite-dimensional algebras tends to privilege Ω_A from a notational point of view, while the one on triangulated categories would rather use Σ_A; we sometimes use both. If A is clear from the context and no confusion arises, we write Ω and Σ instead of Ω_A and Σ_A, respectively. If B is a subalgebra of A we have a canonical isomorphism $\Omega_A^n(A \otimes_B V) \cong A \otimes_B \Omega_B^n(V)$ in $\underline{\mathrm{mod}}(A)$, where V is a finitely generated B-module and n a positive integer. If moreover A is projective as B-module, we also have a canonical isomorphism $\Omega_B^n(\mathrm{Res}_B^A(U)) \cong \mathrm{Res}_B^A(\Omega_A^n(U))$ in $\underline{\mathrm{mod}}(B)$, where U is a finitely generated A-module, and the canonical adjunction isomorphism $\mathrm{Hom}_A(A \otimes_B V, U) \cong \mathrm{Hom}_B(B, \mathrm{Res}_B^A(U))$ induces an isomorphism $\underline{\mathrm{Hom}}_A(A \otimes_B V, U) \cong \underline{\mathrm{Hom}}_B(V, \mathrm{Res}_B^A(U))$, where U is an A-module and V is a B-module. The following well-known lemma states that the Heller translates commute with the adjunction isomorphisms (we sketch a proof for the convenience of the reader).

Lemma 2.1. *Let A be a finite-dimensional algebra over a field k and B a subalgebra such that A is projective as B-module. Let n be a positive integer. The following diagram of canonical isomorphisms is commutative:*

$$
\begin{array}{ccc}
\underline{\mathrm{Hom}}_A(A \otimes_B \Omega_B^n(V), \Omega_A(U)) & \xrightarrow{\simeq} & \underline{\mathrm{Hom}}_B(\Omega_B^n(V), \mathrm{Res}_B^A(\Omega_A^n(U))) \\
\simeq \downarrow & & \downarrow \simeq \\
\underline{\mathrm{Hom}}_A(\Omega_A^n(A \otimes_B V), \Omega_A^n(U)) & & \underline{\mathrm{Hom}}_B(\Omega_B^n(V), \Omega_B^n(\mathrm{Res}_B^A(U))) \\
\Omega_A^n \uparrow \simeq & & \simeq \uparrow \Omega_B^n \\
\underline{\mathrm{Hom}}_A(A \otimes_B V, U) & \xrightarrow{\simeq} & \underline{\mathrm{Hom}}_B(V, \mathrm{Res}_B^A(U))
\end{array}
$$

Proof. We sketch the argument in the case $n = 1$; the general case follows either by induction, or directly with short exact sequences in the two diagrams below replaced by exact sequences with $n+2$ terms of which all but possibly the first and last are projective. The Heller translates Ω_A and Ω_B are defined by the following two commutative diagrams with exact rows, where P_X denotes a projective cover of the module X:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes_B \Omega_B(V) & \longrightarrow & A \otimes_B P_V & \xrightarrow{\nu} & A \otimes_B V & \longrightarrow & 0 \\
& & \simeq \downarrow & & \delta \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_A(A \otimes_B V) & \longrightarrow & P_{A \otimes_B V} & \longrightarrow & A \otimes_B V & \longrightarrow & 0 \\
& & \Omega_\alpha \downarrow & & \gamma \downarrow & & \alpha \downarrow & & \\
0 & \longrightarrow & \Omega_A(U) & \longrightarrow & P_U & \xrightarrow{\mu} & U & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_B(V) & \longrightarrow & P_V & \xrightarrow{\rho} & V & \longrightarrow & 0 \\
& & \Omega_\beta \downarrow & & \phi \downarrow & & \beta \downarrow & & \\
0 & \longrightarrow & \Omega_B(\mathrm{Res}_B^A(U)) & \longrightarrow & P_{\mathrm{Res}_B^A(U)} & \longrightarrow & \mathrm{Res}_B^A(U) & \longrightarrow & 0 \\
& & \simeq \downarrow & & \psi \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Res}_B^A(\Omega_A U) & \longrightarrow & \mathrm{Res}_B^A(P_U) & \xrightarrow{\sigma} & \mathrm{Res}_B^A(U) & \longrightarrow & 0
\end{array}
$$

Here μ, ρ are chosen surjective morphisms and $\sigma = \mathrm{Res}_B^A(\mu)$, $\nu = \mathrm{Id}_A \otimes \rho$. We have that $\mu \circ \gamma \circ \delta = \alpha \circ \nu$. Since α, β correspond to each other through the adjunction ismorphism, the two diagrams imply that the images of Ω_α and Ω_β correspond to each other through the adjunction $\underline{\mathrm{Hom}}_B(\Omega_B(V), \mathrm{Res}_B^A(\Omega_A(U))) \cong \underline{\mathrm{Hom}}_A(A \otimes_B \Omega_B(V), \Omega_A(U))$. □

A morphism $\alpha : U \to V$ in $\underline{\mathrm{mod}}(A)$ is called *almost vanishing* if $\alpha \neq 0$ and if for any morphism $\varphi : X \to U$ in $\underline{\mathrm{mod}}(A)$ which is not a split epimorphism we have $\alpha \circ \varphi = 0$. Since endomorphism algebras of indecomposable A-modules are (not necessarily split) local, it follows from [5, I.4.1] that this condition is equivalent to its dual; that is, for any morphism $\psi : V \to Y$ in $\underline{\mathrm{mod}}(A)$ which is not a split monomorphism we have $\psi \circ \alpha = 0$. In order to follow almost vanishing morphisms through the standard adjunction isomorphisms we collect a few elementary statements on split epimorphisms. In statements and proofs where we consider both A-homomorphisms and their classes in the stable category we adopt the notational convention that if $\varphi : U \to V$ is a homomorphism of A-modules then $\underline{\varphi}$ denotes the class of φ in $\underline{\mathrm{Hom}}_A(U, V)$. As in any triangulated category, epimorphisms in $\underline{\mathrm{mod}}(A)$ are split epimorphisms, and those are essentially induced by split epimorphisms in $\mathrm{mod}(A)$:

Lemma 2.2. *Let A be a finite-dimensional k-algebra and $\varphi : U \to V$ an A-homomorphism. Suppose that V is indecomposable non projective. Then φ is a split epimorphism in $\mathrm{mod}(A)$ if and only if its class $\underline{\varphi} : U \to V$ is a split epimorphism in $\underline{\mathrm{mod}}(A)$.*

Proof. Suppose $\underline{\varphi}$ is a split epimorphism in $\underline{\mathrm{mod}}(A)$. Let $\sigma : V \to U$ be an A-homomorphism such that $\underline{\varphi} \circ \underline{\sigma} = \underline{\mathrm{Id}}_V$. Thus $\varphi \circ \sigma - \mathrm{Id}_V$ factors through a projective module. Since V is indecomposable non projective this implies that $\varphi \circ \sigma - \mathrm{Id}_V \in J(\mathrm{End}_A(V))$, hence $\rho = \varphi \circ \sigma$ is an automorphism of V and $\sigma \circ \rho^{-1}$ a section for φ. The converse is trivial. □

Lemma 2.3. *Let A be a k-algebra and B a subalgebra of A. Suppose that B has a complement in A as a B-B-bimodule. Let $\varphi : V \to W$ be a homomorphism of left B-modules and set $\psi = \mathrm{Id}_A \otimes \varphi : A \otimes_B V \to A \otimes_B W$. Then ψ is a split epimorphism if and only of φ is a split epimorphism.*

Proof. Suppose that ψ is a split epimorphism; that is, there is an A-homomorphism $\sigma : A \otimes_B W \to A \otimes_B V$ satisfying $\psi \circ \sigma = \mathrm{Id}_{A \otimes_B W}$. Let C be a complement of B in A as a B-B-bimodule; that is, $A = B \oplus C$ as a B-B-bimodule. Denote by $\pi : A \otimes_B V \to V$ and $\tau : A \otimes_B W \to W$ the canonical projections with kernel $C \otimes_B V$ and $C \otimes_B W$, respectively, induced by the projection $A \to B$ with kernel C. Explicitly, for $v \in V$ and $a \in A$ we have $\pi(a \otimes v) = a \otimes v$ if $a \in B$ and $\pi(a \otimes v) = 0$ if $a \in C$; similarly for τ. Since $\psi = \mathrm{Id}_A \otimes \varphi$ we have $\varphi \circ \pi = \tau \circ \psi$. Define $\rho : W \to V$ by $\rho(w) = \pi(\sigma(1 \otimes w))$, for any $w \in W$. Then $\varphi(\rho(w)) = \varphi(\pi(\sigma(1 \otimes w))) = \tau(\psi(\sigma(1 \otimes w))) = \tau(1 \otimes w) = w$, hence φ is a split epimorphism with section ρ. The converse is trivial. □

Lemma 2.4. *Let A be a finite-dimensional algebra over an algebraically closed field k and B a subalgebra such that A is projective as left and right B-module and such that B has a complement in A as a B-B-bimodule. Let V be a finitely generated indecomposable non projective B-module such that $U = A \otimes_B V$ is indecomposable. Then U is non projective. Let $\zeta_V : V \to \Omega_B(V)$ and $\zeta_U : U \to \Omega_A(U)$ be almost vanishing homomorphisms in $\underline{\mathrm{mod}}(B)$ and $\underline{\mathrm{mod}}(A)$, respectively. Suppose that there is an isomorphism $\beta : \Omega_B^n(V) \cong V$ in $\underline{\mathrm{mod}}(B)$ such that $\Omega_B(\beta) \circ \Omega_B^n(\zeta_V) = (-1)^n \zeta_V \circ \beta$. Then $\Omega_A^n(U) \cong U$ and for any isomorphism $\alpha : \Omega_A^n(U) \cong U$ in $\underline{\mathrm{mod}}(A)$ we have the $\Omega_A(\alpha) \circ \Omega_A^n(\zeta_U) = (-1)^n \zeta_U \circ \alpha$.*

Proof. We first point out where we use the hypothesis on k being algebraically closed. If we can find *some* isomorphism $\alpha : \Omega_A^n(U) \cong U$ in $\underline{\mathrm{mod}}(A)$ such that $\Omega_A(\alpha) \circ \Omega_A^n(\zeta_U) = (-1)^n \zeta_U \circ \alpha$ then this holds for *any* isomorphism because $\mathrm{End}_A(U)$ is split local. Moreover, ζ_U is unique up to a nonzero scalar. Similarly for V and ζ_V. By the assumptions, $A = B \oplus C$ for some B-B-submodule C of A which is finitely generated projective as left and right B-module. Thus Res_B^A sends any projective A-module to a projective B-module, and V is a direct summand of $\mathrm{Res}_B^A(U)$. Since V is non projective this implies that U is non projective. By adjunction we have

$$\underline{\mathrm{Hom}}_A(U, U) \cong \underline{\mathrm{Hom}}_B(V, \mathrm{Res}_B^A(A \otimes_B V))$$

We prove that the image η of ζ_U under this adjunction is a morphism which, when composed with any projection of $\mathrm{Res}_B^A(A \otimes_B V)$ onto an indecomposable direct summand is either zero or almost vanishing in $\underline{\mathrm{mod}}(B)$. This is equivalent to showing that η precomposed with any morphism ending at V which is not a split epimorphism in $\underline{\mathrm{mod}}(B)$ yields zero. Let X be a B-module and $\varphi \in \underline{\mathrm{Hom}}_B(X, V)$ such that the image $\underline{\varphi}$ in $\underline{\mathrm{Hom}}_B(X, V)$ is not a split epimorphism. Then by Lemma 2.2 and Lemma 2.3 the morphism $\psi = \mathrm{Id}_A \otimes \varphi$ in $\underline{\mathrm{Hom}}_A(A \otimes_B X, U)$ is not a split epimorphism and hence $\zeta_M \circ \psi$ is zero in $\underline{\mathrm{Hom}}_A(A \otimes_B X, M)$. Applying the adjunction to $\zeta_M \circ \psi$ we get that $\eta \circ \varphi$ is zero in $\underline{\mathrm{Hom}}_B(X, \mathrm{Res}_B^A(A \otimes_B V)))$. Thus all components of η to the indecomposable direct summands of $\mathrm{Res}_B^A(A \otimes_B V))$ are either zero or a scalar multiple of ζ_V. The hypothesis on ζ_V implies that $\Omega_B^n(\eta) = (-1)^n \eta$, modulo appropriate identifications. It follows from Lemma 2.1 that $\Omega^n(\zeta_U) = (-1)^n \zeta_U$, again modulo appropriate identifications, which proves the result. $\qquad\square$

The existence of an almost vanishing morphism $\alpha : U \to V$ in $\underline{\mathrm{mod}}(A)$ forces that U, V are indecomposable non projective A-modules, and that the exact triangle in $\underline{\mathrm{mod}}(A)$ of the form

$$\Omega(V) \longrightarrow E \longrightarrow U \xrightarrow{\ \alpha\ } V$$

is *almost split* in the sense of [5, I.4.1.], hence induced by an almost split exact sequence of A-modules of the form

$$0 \longrightarrow \Omega(V) \longrightarrow E \longrightarrow U \longrightarrow 0$$

which in turn forces that $V \cong \Omega(U)$ as A is symmetric (cf. [2, 4.12.8]). This shows that if there is an almost vanishing morphism $U \to \Sigma^n(U)$ in $\underline{\mathrm{mod}}(A)$ for some integer n then $\Sigma^n(U) \cong \Sigma^{-1}(U)$, and hence either $n = -1$ or U is periodic of period dividing $n + 1$. As mentioned earlier, almost vanishing morphisms define elements in the graded center. It is convenient to extend the terminology of almost vanishing morphisms to elements in the graded center:

Definition 2.5. Let A be a symmetric algebra over a field k. For any integer n we say that an element $\varphi \in Z^n(\underline{\mathrm{mod}}(A))$ is *almost vanishing* if for any indecomposable non projective A-module U either $\varphi(U) = 0$ or $\varphi(U) : U \to \Sigma^n(U)$ is almost vanishing. An element in $Z^*(\underline{\mathrm{mod}}(A))$ is called *almost vanishing* if all of its homogeneous components are almost vanishing. We denote by $\mathcal{I}_A$ the set of all almost vanishing elements in $Z^*(\underline{\mathrm{mod}}(A))$.

This definition makes sense for arbitrary triangulated categories. Note that the zero element of $Z^*(\underline{\mathrm{mod}}(A))$ is contained in $\mathcal{I}_A$; this slight departure from the definition of almost vanishing morphisms has the following obvious advantage:

Lemma 2.6. *Let A be a symmetric algebra over a field k. Then $\mathcal{I}_A$ is an ideal in $Z^*(\underline{\mathrm{mod}}(A))$ whose square is zero. Moreover, $\mathcal{I}_A$ annihilates every ideal consisting of nilpotent elements in $Z^*(\underline{\mathrm{mod}}(A))$.*

Proof. If $\varphi \in Z^n(\underline{\mathrm{mod}}(A))$ is nilpotent, where n is an integer, then for any indecomposable non projective A-module U the morphism $\varphi(U) : U \to \Sigma^n(U)$ is not an isomorphism. Since U, $\Sigma^n(U)$ are both indecomposable, $\varphi(U)$ is neither a split epimorphism nor a split monomorphism. Thus $\varphi(U)$ composed with any almost vanishing morphism yields zero. The result follows. $\square$

In the statements of Theorem 1.2 and Theorem 1.3, in order to show that $Z^*(\underline{\mathrm{mod}}(A))$ is not finite-dimensional in a particular degree we actually show that $\mathcal{I}_A$ is not finite-dimensional in that degree. We will need the following lemma, which is well-known and strictly analogous to [10, Proposition 1.3], for instance.

Lemma 2.7. *Let p be a prime, P a finite p-group and k a field of characteristic p. Evaluation at the trivial kP-module k induces a surjective graded k-algebra homomorphism $Z^*(kP) \to \hat{H}^*(P;k)$ whose kernel $\mathcal{I}$ is a nilpotent ideal.*

Proof. This is the same proof as that of [10, Proposition 1.3], with $H^*(P;k)$, $HH^*(kP)$, $D^b(kP)$ replaced by Tate cohomology $\hat{H}^*(P;k)$, the Tate analogue of Hochschild cohomology $\hat{HH}^*(kP)$ and $\underline{\mathrm{mod}}(kP)$, respectively. $\square$

3. On some modules of period 1 and 2

In order to construct elements in a given degree n of the graded center of the stable category of a symmetric algebra A over a field k it is not sufficient to construct A-modules with period dividing $n+1$; we also need to have some control over the effect of the shift on endomorphisms in order to verify the additional compatibility condition with the shift functor of natural transformations belonging to $Z^n(\underline{\mathrm{mod}}(A))$. This is the purpose of the elementary observations in this section. We use without further comment that if k has positive characteristic p and if $x \in Z(A)$ such that $x^{p-1} \neq 0$, $x^p = 0$, then $1 + x$ is an invertible element in A of order p and generates a subalgebra which can be identified with the truncated polynomial algebra $k[x]/(x^p)$ or the group algebra $k\langle 1 + x\rangle$, whichever is more convenient. The algebra A may or may not be projective as $k\langle 1 + x\rangle$-module; this will depend on x. If $A = kP$ is the group algebra of an elementary abelian p-group P with minimal generating set $\{u_1, u_2, .., u_r\}$, then any nonzero k-linear combination $x = \sum_{i=1}^{r} \lambda_i(1 - u_i)$ has the property that A is projective as $k\langle 1 + x\rangle$-module. The subgroups generated by $1 + x$ for those x are called cyclic shifted subgroups of A.

Lemma 3.1. *Let A be a finite-dimensional algebra over a field k of prime characteristic p and x an element in A such that $x^p = 0$, $x^{p-1} \neq 0$ and such that A is projective as a $k\langle 1 + x\rangle$-module. Let $u \in A^\times$ such that $ux = xu$. Then $(ux)^p = 0$, $(ux)^{p-1} \neq 0$ and A is projective as a $k\langle 1 + ux\rangle$-module.*

Proof. Since x and u commute we clearly have $(ux)^p = 0$ and $(ux)^{p-1} \neq 0$. The hypothesis that A is projective as $k\langle 1 + x\rangle$-module is equivalent to $\dim_k(x^{p-1}A) = \frac{1}{p}\dim_k(A)$. As u is invertible and commutes to x, we have $(ux)^{p-1}A = x^{p-1}A$, and hence A is projective as $k\langle 1 + ux\rangle$-module. $\square$

Lemma 3.2. *Let A be a finite-dimensional algebra over a field k of prime characteristic $p > 0$. Let $x \in A$ such that $x^{p-1} \neq 0$, $x^p = 0$ and such that A is free as right $k\langle 1 + x \rangle$-module. Set $M = Ax^{p-1}$.*

(i) If $p = 2$ there is an isomorphism $\alpha : M \cong \Omega(M)$ in $\underline{\mathrm{mod}}(A)$ such that $\underline{\varphi} \circ \alpha = \alpha \circ \Omega(\underline{\varphi})$ for any $\varphi \in \mathrm{End}_A(M)$.

(ii) If $p > 2$ there is an isomorphism $\beta : M \cong \Omega^2(M)$ in $\underline{\mathrm{mod}}(A)$ such that $\underline{\varphi} \circ \beta = \beta \circ \Omega^2(\underline{\varphi})$ for any $\varphi \in \mathrm{End}_A(M)$.

(iii) We have $\underline{\mathrm{End}}_A(M) \cong \mathrm{End}_A(M)$.

Proof. Since A is free as right $k\langle 1 + x \rangle$-module we have an exact sequence of A-modules of the form

$$0 \longrightarrow M \longrightarrow A \overset{\epsilon}{\longrightarrow} A \overset{\pi}{\longrightarrow} M \longrightarrow 0$$

where ϵ is given by right multiplication with x and π is given by right multiplication with x^{p-1}. Thus for this choice of projective covers we have $\Omega(M) = Ax$ and $\Omega^2(M) = M$. Let $\varphi \in \mathrm{End}_A(M)$. Then φ lifts to an endomorphism $\psi \in \mathrm{End}_A(A)$ satisfying $\pi \circ \psi = \varphi \circ \pi$. Let $y \in A$ such that $\psi(a) = ay$ for all $a \in A$. Thus $\varphi(ax^{p-1}) = ayx^{p-1} = ax^{p-1}y$ for all $a \in A$. Also, since x is central we have $\epsilon \circ \psi = \psi \circ \epsilon$. Thus ψ restricts to the endomorphism φ of M, which shows (i) and (ii). Suppose now that φ factors through a projective A-module. Then φ factors through the inclusion $M \subseteq A$, or equivalently, there is $y \in A$ such that $\varphi(m) = my$ for all $m \in M$ and such that $Ay \subseteq M$. But then in particular $y \in M = Ax^{p-1}$, hence $y = ux^{p-1}$ for some $u \in A$. Since any $m \in M$ is of the form $m = ax^{p-1}$ for some $a \in A$ we get that $\varphi(m) = \varphi(ax^{p-1}) = ax^{p-1}ux^{p-1} = aux^{2(p-1)} = 0$, whence (iii). $\square$

The next observation shows that if the center of a finite p-group P has rank at least 2 then the construction principle for modules of period 1 and 2 over the group algebra $A = kP$ in the two previous lemmas yields infinitely many isomorphism classes whenever the field k is infinite.

Proposition 3.3. *Let p be a prime, P a finite p-group and k a field of characteristic p. Set $A = kP$. Suppose that $Z(P)$ has an elementary abelian subgroup $E = \langle s \rangle \times \langle t \rangle$ of rank 2, for some elements $s, t \in Z(P)$ of order p. Let $(\lambda, \mu), (\lambda', \mu') \in k^2 - \{(0,0)\}$. Set $x = \lambda(s-1) + \mu(t-1)$ and $x' = \lambda'(s-1) + \mu'(t-1)$. Then Ax is absolutely indecomposable as left A-module, and the following statements are equivalent.*

(i) $Ax \cong Ax'$ as left A-modules.

(ii) $Ax = Ax'$.

(iii) The images of (λ, μ) and (λ', μ') in $\mathbb{P}^1(k)$ are equal.

Proof. Since $A = kP$ is split local, any quotient of A as A-module is absolutely indecomposable. Suppose (i) holds. Since A is symmetric, any isomorphism $Ax \cong Ax'$ extends to an endomorphism of A, which is given by right multiplication with an element $y \in A$. In particular, $Axy = Ax'$. But $x \in Z(A)$ implies $Axy = Ayx \subseteq Ax$. Thus $Ax' \subseteq Ax$. Exchanging the roles of x and x' shows $Ax = Ax'$, whence (ii) holds. The converse is trivial. Suppose now that (ii) holds. Arguing by contradiction, suppose that the images of $(\lambda, \mu), (\lambda', \mu')$ in $\mathbb{P}^1(k)$ are different, or equivalently, that $(\lambda, \mu), (\lambda', \mu')$ are linearly independent in k^2. Since $Ax = Ax'$ contains any k-linear combination of x and x' it follows that Ax contains any k-linear combination of $s - 1$ and $t - 1$. But then $A(s-1) = Ax = A(t-1)$ because all involved modules have the same dimension $\frac{p-1}{p} \dim_k(A)$ as A is projective as right kE-module. However, clearly

$kE(s-1) \neq kE(t-1)$, so this is impossible. This shows that (ii) implies (iii). The converse is again trivial. $\square$

Corollary 3.4. *Let p be a prime, P a finite p-group of rank at least 2 and k an infinite field of characteristic p. If $p = 2$ then kP has infinitely many isomorphism classes of absolutely indecomposable modules of dimension $\frac{|P|}{2}$ whose period is 1, and if $p > 2$ then kP has infinitely many isomorphism classes of absolutely indecomposable modules of dimension $\frac{|P|}{p}$ whose period is 2.*

Proof. Let E be an elementary abelian subgroup of rank 2 of P. By Proposition 3.3 and Lemma 3.2, the result holds for kE instead of kP. Induction Ind_E^P is an exact functor, preserving periodicity, and by Mackey's formula, for any indecomposable kP-module M there are, up to isomorphism, only finitely many indecomposable kE-modules N satisfying $\mathrm{Ind}_E^P(N) \cong M$, which implies the statement. $\square$

4. Proof of Theorem 1.1 and Theorem 1.2

Let p be a prime and P a finite p-group of rank at least 2. Let k be an algebraically closed field of characteristic p. By Lemma 2.7, evaluation at the trivial kP-module induces a surjective graded algebra homomorphism $Z^*(\underline{\mathrm{mod}}(kP)) \to \hat{H}^*(P; k)$ with nilpotent kernel $\mathcal{I}$. Let $E = \langle s \rangle \times \langle t \rangle$ be an elementary abelian subgroup of $Z(P)$ of rank 2 for some elements s, t in $Z(P)$ of order P and let $(\lambda, \mu) \in k^2 - \{(0,0)\}$ Set $x = \lambda(s-1) + \mu(t-1)$ and $M = Ax^{p-1}$. By Lemma 3.2 the module M has period 1 if $p = 2$ and period 2 if p is odd. The almost split exact sequence ending in M is of the form

$$0 \longrightarrow \Omega^2(M) \longrightarrow E \longrightarrow M \longrightarrow 0$$

hence represented by an almost vanishing morphism $\zeta_M : M \to \Omega(M)$. By [10, Proposition 1.4], this defines a natural transformation $\varphi_M : \mathrm{Id} \to \Omega$ as follows. Set $\varphi_M(M) = \zeta_M$, set $\varphi_M(N) = 0$ if N is indecomposable non projective and not isomorphic to M or $\Omega(M)$, and in the case p odd, set $\varphi_M(\Omega(M)) = -\Omega(\zeta_M)$. Then, thanks to Lemma 3.2 again, φ is indeed a natural transformation belonging to $Z^{-1}(\underline{\mathrm{mod}}(kP))$. If $p = 2$ (resp. $p > 2$)then for any integer n (resp. even integer n) we have $M \cong \Sigma^n(M)$, and hence φ determines also an element in $Z^{n-1}(\underline{\mathrm{mod}}(kP))$ for any integer n (resp. even integer n). By Lemma 3.3, this implies that $Z^{n-1}(\underline{\mathrm{mod}}(kP))$ is infinite dimensional for any integer n if $p = 2$ and for any even integer if $p > 2$. This proves Theorem 1.1 and Theorem 1.2.

Remark 4.1. D. J. Benson [3] pointed out that it is easy to construct indecomposable modules for finite p-group algebras with period one for odd p. One might therefore wonder whether the stronger statement of Theorem 1.1 holds for p odd as well.

5. Proof of Theorem 1.3

Let k be an algebraically closed field of characteristic 2. Denote by $V_4 = \langle s, t \rangle$ a Klein four group, with commuting involutions s, t. By results of Bašev [1], Heller and Reiner [6] (see also [2, 4.3.3] for an expository account), for any positive integer n and any $\lambda \in k \cup \{\infty\}$ there is, up to isomorphism, a unique indecomposable kV_4-module M_n^λ of dimension $2n$, having a k-basis $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ on which the elements $x = s - 1$ and $y = t - 1$ act by

$$xa_i = b_i \ (1 \le i \le n), \ ya_i = \lambda b_i + b_{i+1} \ (1 \le i < n), \ ya_n = \lambda b_n \ , xb_i = 0 = yb_i \ (1 \le i \le n)$$

provided that $\lambda \in k$, and by

$$xa_i = b_{i+1} \ (1 \le i < n), \ xa_n = 0, \ ya_i = b_i \ (1 \le i \le n), \ ya_n = b_n \ , xb_i = 0 = yb_i \ (1 \le i \le n)$$

if $\lambda = \infty$. Then M_n^λ, with $\lambda \in k \cup \{\infty\}$, is a set of representatives of the isomorphism classes of indecomposable kV_4-modules of dimension $2n$. This set can be indexed by the elements in $\mathbb{P}^1(k)$ with M_n^λ corresponding to the image of $(\lambda, 1)$ in $\mathbb{P}^1(k)$ if $\lambda \in k$, and with M_n^∞ corresponding to the image of $(0, 1)$ in $\mathbb{P}^1(k)$. The natural action of $GL_2(k)$ on $\mathbb{P}^1(k)$ translates to a transitive action of $GL_2(k)$ on the set of isomorphism classes of $2n$-dimensional indecomposable kV_4-modules. The group $GL_2(k)$ can be identified with a group of k-algebra automorphisms of kV_4, with $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ acting on kV_4 by $g(x) = \alpha x + \beta y$ and $g(y) = \gamma x + \delta y$. In this way, any kV_4-module M gives rise via restriction along the automorphism g to a kV_4-module ${}^g M$, which is equal to M as k-vector space, with $z \in kV_4$ acting as $g(z)$ on M. This defines another action of $GL_2(k)$ on the set of isomorphism classes of $2n$-dimensional indecomposable kV_4-modules, and, of course, these actions are compatible: one checks that for $\lambda \in k$ we have $M_n^\lambda \cong {}^g(M_n^0)$, where $g = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$, and that $M_n^\infty \cong {}^h(M_n^0)$, where $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The point of these remarks is that in order to calculate (stable) endomorphism algebras of the modules M_n^λ, identify almost vanishing morphisms and check their invariance under the Heller operator, it suffices to consider the case $\lambda = 0$. We collect some basic facts on endomorphisms of the modules M_n^λ, many of which are, of course, well-known (such as the fact that M_n^λ has period 1). In what follows we set $M_n = M_n^0$, for any positive integer n.

Lemma 5.1. *Let n be a positive integer and denote by $\{a_1, a_2, .., a_n, b_1, b_2, .., b_n\}$ a k-basis of M_n such that $xa_i = b_i$ for $1 \le i \le n$, $ya_i = b_{i+1}$ for $1 \le i < n$, $ya_n = 0$ and $xb_i = 0 = yb_i$ for $1 \le i \le n$. Let $\varphi \in \mathrm{End}_{kV_4}(M_n)$.*

(i) We have $\Omega(M_n) \cong M_n$.

(ii) If $\varphi \in \mathrm{End}^{pr}_{kV_4}(M_n)$ then $\mathrm{Im}(\varphi) \subseteq \mathrm{soc}(M_n)$.

(iii) If $\mathrm{Im}(\varphi) = k \cdot b_1$ then $\varphi \notin \mathrm{End}^{pr}_{kV_4}(M_n)$.

(iv) For $1 \le i \le n$ denote by φ_i^n the endomorphism of M_n sending a_i to b_1 and all other basis elements of M_n to zero. Then the set $\{\underline{\varphi}_i^n\}_{1 \le i \le n}$ is linearly independent in $\underline{\mathrm{End}}_{kV_4}(M_n)$.

(v) We have $\dim_k(\mathrm{End}^{pr}_{kV_4}(M_n)) = n^2 - n$.

Proof. For $1 \le i \le n$ denote by s_i the element in $(kV_4)^n$ whose i-th component is 1 and whose other componenets are 0. Then $\{s_1, s_2, .., s_n\}$ is a basis of the free kV_4-module $(kV_4)^n$. There is an injective kV_4-homomorphism

$$\iota : M_n \longrightarrow (kV_4)^n$$

such that $\iota(a_i) = ys_i + xs_{i+1}$ for $1 \le i < n$ and $\iota(a_n) = ys_n$, and there is a surjective kV_4-homomorphism

$$\pi : (kV_4)^n \longrightarrow M_n$$

such that $\pi(s_i) = a_i$ for $1 \le i \le n$. One checks that $\mathrm{Im}(\iota) = \ker(\pi)$, which shows that $\Omega(M_n) \cong M_n$, whence (i). If $\varphi \in \mathrm{End}^{pr}_{kV_4}(M_n)$, the inclusion $\mathrm{Im}(\varphi) \subset \mathrm{rad}(M_n)$ is a general fact, and we have $\mathrm{rad}(M_n) = \mathrm{soc}(M_n)$, which shows (ii). Since φ factors through a projective module, it factors through both ι and π; thus there is $\psi \in \mathrm{End}_{kV_4}((kV_4)^n)$ such that $\varphi = \pi \circ \psi \circ \iota$.

For $1 \leq i, j \leq n$ let $\alpha_{i,j} \in k$ and $r_{i,j} \in \mathrm{rad}(kV_4) = xkV_4 + ykV_4$, viewed as element in the j-th component of $(kV_4)^n$, such that

$$\psi(s_i) = \sum_{j=1}^{n} \alpha_{i,j} \ s_j \ + \ r_{i,j}$$

Since $\mathrm{Im}(\iota) \subset \mathrm{rad}((kV_4)^n)$ one easily sees that the we may assume $r_{i,j} = 0$. Two short verifications show that

$$\varphi(a_n) = \sum_{j=2}^{n} \alpha_{n,j-1} \ b_j$$

and, for $1 \leq i < n$,

$$\varphi(a_i) = \alpha_{i+1,1}b_1 \ + \ \sum_{j=2}^{n} (\alpha_{i,j-1} + \alpha_{i+1,j})b_j$$

which also shows again that $\mathrm{Im}(\varphi) \subseteq \mathrm{soc}(M_n)$. Moreover, this calculation shows that if $\mathrm{Im}(\varphi) \subseteq kb_1$, then the equation for $\varphi(a_n)$ implies $\alpha_{n,j} = 0$ for $1 \leq j < n$, and the equations for the $\varphi(a_i)$ then imply inductively that $\alpha_{i,j} = 0$ for $1 \leq j < i \leq n$, which in turn yields $\varphi = 0$. This proves (iii). Statement (iv) is an immediate consequence of (iii). For any pair (i,j), with $1 \leq i, j \leq n$ there is a unique endomorphism of M_n sending a_i to b_j and all other basis elements to zero. Thus the space of endomorphisms of M_n whose image is contained in $\mathrm{soc}(M_n)$ has dimension n^2. It follows form (iv) that $\dim_k(\mathrm{End}^{pr}_{kV_4}(M_n)) \leq n^2 - n$. Setting $\varphi_{i,j} = \pi \circ \psi_{i,j} \circ \iota$, where $\psi_{i,j}$ is the unique endomorphism determined by the coefficients $\alpha_{i,j} = 1$ and $\alpha_{r,s} = 0$ for $(r,s) \neq (i,j)$ an easy verification shows that the set $\{\varphi_{i,j} \mid 1 \leq i \leq n, \ 1 \leq j < n\}$ is linearly independent in $\mathrm{End}^{pr}_{kV_4}(M_n)$, whence the equality in (v). $\qquad\square$

Lemma 5.2. *For any positive integer n denote by $\{\alpha_1^n, \alpha_2^n, .., \alpha_n^n, \beta_1^n, \beta_2^n, .., \beta_n^n\}$ a k-basis of M_n such that $xa_i^n = b_i^n$ for $1 \leq i \leq n$, $ya_i^n = b_{i+1}^n$ for $1 \leq i \leq n-1$, $ya_n^n = 0$, and such that $xb_i^n = 0 = yb_i^n$ for $1 \leq i \leq n$. For any positive integers $m < n$ there is a homomorphism $\zeta_m^n : M_m \rightarrow M_n$ such that $\zeta(a_i^m) = a_{n-m+i}^n$ for $1 \leq i \leq m$, and a homomorphism $\xi_m^n : M_n \rightarrow M_m$ such that $\xi_m^n(a_i^n) = a_i^m$ for $1 \leq i \leq m$ and $\xi_m^n(a_i^n) = 0$ for $m+1 \leq i \leq n$. Then the sequence*

$$0 \longrightarrow M_m \xrightarrow{\ \zeta_m^n\ } M_n \xrightarrow{\ \xi_{n-m}^n\ } M_{n-m} \longrightarrow 0$$

is exact. In particular ζ_m^n is a non-split monomorphism and ξ_m^n is a non-split epimorphism.

Proof. Straightforward verification. $\qquad\square$

Lemma 5.3. *Let n be a positive integer. The following hold.*

(i) The endomorphism $\underline{\varphi}_1^n$ in $\underline{\mathrm{End}}_{kV_4}(M_n)$ is an almost vanishing morphism.

(ii) For any isomorphism $\alpha : M_n \cong \Omega(M_n)$ in $\underline{\mathrm{mod}}(kV_4)$ We have $\alpha \circ \underline{\varphi}_1^n = \Omega(\underline{\varphi}_1^n) \circ \alpha$.

Proof. Since M_n has period 1, any almost vanishing morphism starting at M_n is an endomorphism of M_n. Let $\psi : M_n \rightarrow M_n$ be a representative of an almost vanishing endomorphism. We use the notation of the previous lemma, except that we drop the superscripts of basis elements. First compose ψ with ζ_n^{n+1}. Given that $\underline{\psi}$ is almost vanishing and ζ_n^{n+1} is not an split monomorphism, $\zeta_n^{n+1} \circ \psi$ has to be a zero morphism in $\underline{\mathrm{mod}}(kV_4)$. But any kV_4-homomorphism between indecomposable non projective modules that factors through a projective module has its image contained in the socle of the target module. As ζ_n^{n+1} is a monomorphism, it implies

that the image of ψ has to be contained in $\mathrm{soc}(M_n)$. Using Lemma 5.1 (iv), ψ can be chosen in the k-vector space generated by $\{\varphi_i^n \,|\, 1 \le i \le n\}$. So there exist $\alpha_i \in k$, $1 \le i \le n$ such that $\psi = \sum_{i=1}^n \alpha_i \varphi_i^n$. The aim is to prove that $\alpha_i = 0$ for all $2 \le i \le n$. To do that we pre-compose ψ with ζ_j^n where $j = n - i + 1$. Again the class of this composition has to be a zero morphism in $\underline{\mathrm{mod}}(kV_4)$. The proof is by reverse induction on i, which amounts to proceeding by induction over $j = n - i + 1$. If $j = 1$ then $\psi \circ \zeta_1^n(a_1) = \alpha_n \varphi_n^n(a_n) = \alpha_n b_1$, where the notation of basis elements of M_n is as at the beginning of the proof of Lemma 5.1. Pre-composing with ξ_1^n one gets an endomorphism $\psi \circ \zeta_1^n \circ \xi_1^n$ of M_n that is equal to $\alpha_n \varphi_1^n$. Since the class of this morphism has to be 0 in $\underline{\mathrm{mod}}(kV_4)$ we get that $\alpha_n = 0$. Suppose now that $\alpha_i = 0$ for all $i > n - j + 1$. Pre-compose ψ with ζ_j^n; as before the class of $\psi \circ \zeta_j^n$ in $\underline{\mathrm{mod}}(kV_4)$ is zero. By direct computation $\psi \circ \zeta_j^n(a_1) = \alpha_{n-j+1} \varphi_{n-j+1}^n(a_{n-j+1}) = \alpha_{n-j+1} b_1$ and $\psi \circ \zeta_j^n(a_i) = \alpha_{n-j+1} \varphi_{n-j+i}^n(a_{n-j+1}) = 0$. Pre-compose now with ξ_j^n and get an endomorphism $\psi \circ \zeta_j^n \circ \xi_j^n$ of M_n that is equal to $\alpha_{n-j+1} \varphi_1^n$. The class of the latter morphism is zero in $\underline{\mathrm{mod}}(kV_4)$ if and only if $\alpha_{n-j+1} = 0$. The induction procedure stops at $j = n - 1$, or equivalently, at $i = 2$, whence (i). In order to prove (ii), note first that of the statement holds for some isomorphism α it holds for any such isomorphism because $\mathrm{End}_{kV_4}(M_n)$ is split slocal and $\underline{\varphi}_1^n$ is almost vanishing. We use the notation introduced at the beginning of the proof of Lemma 5.1; in particular, we have an injective homomorphism $\iota : M_n \to (kV_4)^n$ and a surjective homomorphism $\pi : (kV_4)^n \to M_n$. Define $\theta : (kV_4)^n \to (kV_4)^n$ by $\theta(s_1) = x s_1$ and $\theta(s_j) = 0$ for $2 \le j \ne n$. A straightforward verification shows that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_n & \overset{\iota}{\longrightarrow} & (kV_4)^n & \overset{\pi}{\longrightarrow} & M_n & \longrightarrow & 0 \\
& & \downarrow{\varphi_1^n} & & \downarrow{\theta} & & \downarrow{\varphi_1^n} & & \\
0 & \longrightarrow & M_n & \overset{\iota}{\longrightarrow} & (kV_4)^n & \overset{\pi}{\longrightarrow} & M_n & \longrightarrow & 0
\end{array}
$$

This shows that for some - and hence any - identification $M_n = \Omega(M_n)$ we have $\Omega(\underline{\varphi}_1^n) = \underline{\varphi}_1^n$, which completes the proof. $\qquad\square$

In conjunction with the remarks at the beginning of this section, this shows that for any even dimensional indecomposable non projective kV_4-module M_n^λ there exists, up to multiplication by a scalar, a unique element of the graded center in any degree that is an almost vanishing endomorphism of M_n^λ and zero on all indecomposable modules not isomorphic to M_n^λ. Now let η be a natural transformation belonging to the kernel $\mathcal{I}$ of the canonical evaluation map $Z^*(\underline{\mathrm{mod}}(kV_4)) \to \hat{H}^*(V_4; k)$. Then η is zero when evaluated at the trivial module, hence zero at any odd dimensional inedcomposable kV_4-module, as these are precisely the Ω-orbit of k, up to isomorphism. In order to conclude the proof of Theorem 1.3 we need to show that for any even-dimensional indecomposable kV_4-module M the evaluation $\eta(M)$ is either zero or almost vanishing. Thanks to the remarks at the beginning of this section, it suffices to show this for the modules M_n, for n a positive integer, and this is what the following lemma does:

Lemma 5.4. *Let $\eta \in \mathcal{I}$ and n a positive integer. Then $\eta(M_n)$ is a scalar multiple of $\underline{\varphi}_1^n$.*

Proof. The proof is by induction on n. For $n = 1$ the assertion is trivial. Suppose that $n > 1$ and that η_{M_l} is a multiple of $\underline{\varphi}_1^l$ for all $1 \le l < n$. By Lemma 5.1, there are coefficients $\alpha_i \in k$ such that $\eta(M_n) = \sum_{i=1}^n \alpha_i \underline{\varphi}_i^n$. We prove that, for any for any k such that $n \ge k > 1$, if

$\eta(M_n) = \sum_{i=1}^{k} \alpha_i \underline{\varphi}_i^n$ then $\alpha_k = 0$. The following diagram in $\underline{\mathrm{mod}}(kV_4)$ is commutative:

$$
\begin{array}{ccc}
M_n & \xrightarrow{\underline{\xi}_{n-k+1}^n} & M_{n-k+1} \\
{\scriptstyle \eta(M_n)}\downarrow & & \downarrow{\scriptstyle \eta(M_{n-k+1})} \\
M_n & \xrightarrow[\underline{\xi}_{n-k+1}^n]{} & M_{n-k+1}
\end{array}
$$

By induction $\eta(M_{n-k+1})$ is an almost vanishing morphism or zero, so on one hand, $\eta(M_{n-k+1}) \circ \underline{\xi}_{n-k+1}^n$ is zero in $\underline{\mathrm{mod}}(kV_4)$. On the other hand, an easy verification shows that

$$\underline{\xi}_{n-k+1}^n \circ \eta(M_n) \circ \underline{\zeta}_{n-k+1}^n \;=\; \alpha_k \underline{\varphi}_1^{n-k+1}$$

This forces $\alpha_k = 0$, whence the result. $\square$

Combining the above lemmas yields a proof of Theorem 1.3.

6. RANK 1

For completeness we include some remarks on the graded center of the stable category of a finite p-group P of rank 1. If k is a field of odd characteristic p then P is cyclic, the algebra kP has finite representation type, and the structure of $\underline{\mathrm{mod}}(kP)$ is calculated in [8] and [7]. If $p = 2$ and Q is a non cyclic finite 2-group of rank 1 then Q is a generalised quaternion group.

Theorem 6.1. *Let k be an algebraically closed field of characteristic 2 and let Q be a generalised quaternion 2-group. The kernel $\mathcal{I}$ of the canonical map $Z^*(\underline{\mathrm{mod}}(kQ)) \longrightarrow \hat{H}^*(Q; k)$ given by evaluation at the trivial kQ-module is infinite dimensional in odd degrees.*

The idea of the proof of Theorem 6.1 is to reduce this to the case where $Q = Q_8$ is quaternion of order 8, and then to show that the 2-dimensional modules over the Klein four quotient group V_4 of Q_8 yield infinitely many isomorphism classes of kQ_8-modules of period 2 with the property that Ω^2 acts trivially on the almost vanishing morphisms starting at these modules; this implies that the almost vanishing morphisms determine infinitely many linearly independent elements in each odd degree of the graded center of the stable module category. In what follows, k is a (not necessarily algebraically closed) field of characteristic 2 and Q_8 a quaternion group of order 8, with generators s, t of order 4 satisfying $s^2 = t^2$ and $tst^{-1} = s^{-1}$. Set $z = s^2$; this is the unique (and hence central) involution of Q_8. We start with a few elementary observations.

Lemma 6.2. *We have $kQ_8(z-1) \subseteq Z(kQ_8)$.*

Proof. We have $s(z-1) = s^3 + s$; this is the conjugacy class sum of s, hence contained in $Z(kQ_8)$. The same argument applies to all elements of order 4 in Q_8, whence the result. $\square$

Lemma 6.3. *Let $(\lambda, \mu) \in k^2$ and set $x = \lambda(s-1) + \mu(t-1)$. Suppose that $\lambda^2 + \lambda\mu + \mu^2 \neq 0$. Then $1 + x$ is invertible of order 4 in $(kQ_8)^\times$, and kQ_8 is projective as left or right $k\langle 1+x \rangle$-module.*

Proof. We have $x^2 = (\lambda^2 + \mu^2)(z-1) + \lambda\mu(s-1)(t-1) + \lambda\mu(t-1)(s-1)$. A short calculation shows that $(s-1)(t-1) + (t-1)(s-1) = st(z-1)$. This is an element in $Z(kQ_8)$, by Lemma 6.2. Thus $x^2 = u(z-1)$, where $u = (\lambda^2 + \mu^2) \cdot 1 + \lambda\mu \cdot (st)$. The hypotheses on λ and μ imply that u is invertible. Thus $\langle 1 + x \rangle$ has order 4 and Lemma 3.1 implies that kQ_8 is projective as $k\langle 1 + x^2 \rangle$ module. Since $\langle 1 + x^2 \rangle$ is the unique subgroup of order 2 of $\langle 1 + x \rangle$ it follows that kQ_8 is projective as left or right $k\langle 1 + x \rangle$-module. $\square$

The hypothesis $\lambda^2 + \lambda\mu + \mu^2 \neq 0$ in the previous lemma means that at leat one of λ, μ is nonzero, and if they are both nonzero then $\frac{\lambda}{\mu}$ is not a cube root of unity.

Proposition 6.4. *Let $(\lambda, \mu) \in k^2$ and set $x = \lambda(s-1)+\mu(t-1)$. Suppose that $\lambda^2+\lambda\mu+\mu^2 \neq 0$. Set $M = kQ_8(z-1)x$ and $N = kQ_8x$. Then M, N are absolutely indecomposable, and the following hold.*

(i) We have isomorphisms $\alpha : M \cong \Omega^2(M)$ and $\beta : \Omega(M) \cong N$ in $\underline{\mathrm{mod}}(kQ_8)$.

(ii) We have $M \subseteq N^{\langle z \rangle} = kQ_8(z-1)$.

(iii) For any $\varphi \in \mathrm{Hom}_{kQ_8}(M, N)$ there is an element $c \in kQ_8$ such that $\varphi(m) = cm$ for all $m \in M$; in particular, $\mathrm{Im}(\varphi) \subseteq M$.

(iv) Set $\gamma = \Omega(\beta) \circ \Omega(\alpha) \circ \beta^{-1} : N \cong \Omega^2(N)$. Then, for any $\varphi \in \mathrm{Hom}_{kQ_8}(M, N)$ we have $\Omega^2(\underline{\varphi}) \circ \alpha = \gamma \circ \underline{\varphi}$.

Proof. The modules M, N are cyclic modules over the split local algebra kQ_8, hence absolutely indecomposable. Since $x^2 = u(z-1)$ for some $u \in (kQ_8)^\times$ we have $x^2(z-1) = 0$. Thus $N = kQ_8x$ is contained in the kernel of the surjective homomorphism $kQ_8 \to M$ given by right multiplication with $(z-1)x$, hence equal to this kernel because both have dimension 6. The same argument shows that the endomorphism $kQ_8 \to kQ_8$ given by right multiplication with x has kernel equal to M. This proves that for this choice of projective covers of M and N we get $\Omega(M) = N$ and $\Omega^2(M) = M$, whence (i). The inclusion $M \subseteq N$ is trivial. Since $z - 1$ annihilates M, we have $M \subseteq N^{\langle z \rangle} \subseteq kQ_8(z-1)$, where the second inclusion uses the fact that kQ_8 is projective as right $k\langle z \rangle$-module. Clearly also $\mathrm{Im}(\varphi) \subseteq N^{\langle x \rangle}$. Now kQ_8 is projective as a right $k\langle 1 + x \rangle$-module by 6.3, thus kQ_8 is free of rank 2 as a $k\langle 1 + x \rangle$-module. It follows that kQ_8x, when viewed as right $k\langle 1 + x \rangle$-module, is isomorhic to the direct sum of two copies of the unique (up to isomorphism) 3-dimensional $k\langle 1 + x \rangle$-module. The restriction of this 3-dimensional module to $\langle 1 + x^2 \rangle$ is of the form $k \oplus k\langle 1 + x^2 \rangle$, so the space annihilated by x^2 has dimension $2 + 2 = 4$. Since $x^2 = u(z-1)$ for some invertible element u it follows that the subspace of N annihilated by $z - 1$ has also dimension 4, hence is equal to $kQ_8(z-1)$. This proves (ii). Let $\varphi \in \mathrm{Hom}_{kQ_8}(M, N)$. It follows from (ii) that $\mathrm{Im}(\varphi) \subseteq kQ_8(z-1)$. Thus we may view φ as a homomorphism from the submodule M of $kQ_8(z-1)$ to $kQ_8(z-1)$. Both modules are annihilated by $z - 1$, so we may view this as a homomorphism of kV_4-modules, where $V_4 = Q_8/\langle z \rangle$. But since M is a submodule of $kQ_8(z-1) \cong kV_4$ as kV_4-modules and since kV_4 is commutative it follows that φ is actually induced by *left* multiplication with an element in kV_4. Pulling this back to kQ_8 shows that φ, as a homomorphism of kQ_8-modules, is induced by left multiplication on M with some element $c \in kQ_8$, composed with the inclusion $M \subseteq N$. This proves (iii). In order to prove (iv), we choose notation such that α, β are equalities (that is, we identify $N = \Omega(M)$ and $M = \Omega^2(M)$ in the obvious way described at the beginning of this proof), and consider a commutative diagram of kQ_8-modules of the form

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \overset{\iota}{\longrightarrow} & kQ_8 & \overset{x}{\longrightarrow} & kQ_8 & \overset{x(z-1)}{\longrightarrow} & M & \longrightarrow & 0 \\
& & \downarrow{\varphi'} & & \downarrow{v} & & \downarrow{w} & & \downarrow{\varphi} & & \\
0 & \longrightarrow & N & \underset{\kappa}{\longrightarrow} & kQ_8 & \underset{x(z-1)}{\longrightarrow} & kQ_8 & \underset{x}{\longrightarrow} & N & \longrightarrow & 0
\end{array}
$$

where ι, κ are the inclusions, v, w are suitable elements in kQ_8 and where an arrow labelled by an element in kQ_8 means the homomorphism induced by right multiplication with that

element. We need to show $\underline{\varphi'} = \underline{\varphi}$. For $a \in kQ_8$, the commutativity of the right square in this diagram means that $\varphi(ax(z-1)) = awx$. The commutativity of the middle square is equivalent to $axw = avx(z-1)$, hence $xw = vx(z-1)$. The commutativity of the left square implies that $\varphi'(ax(z-1)) = ax(z-1)v = avx(z-1) = axw$, where we used that $x(z-1) \in Z(kQ_8)$ by Lemma 6.2. In order to show that $\varphi = \varphi'$ it suffices to show that w can be chosen in $Z(kQ_8)$. By (iii) there is an element $c \in kQ_8$ such that $\varphi'(x(z-1)) = cx(z-1) = xc(z-1)$, and hence we may take $w = c(z-1)$, which is an element in $Z(kQ_8)$ by Lemma 6.2. $\qquad\square$

Proof of Theorem 6.1. We use the notation and assumptions of Theorem 6.1; in particular, k is now algebraically closed. We consider first the case where $Q = Q_8$. It follows from Proposition 3.3 applied to the Klein four group $V_4 = Q_8/\langle z \rangle$ and from Proposition 6.4 (i) that if $(\lambda, \mu) \in k^2$ runs over a set of representatives of elements in $\mathbb{P}^1(k)$ satisfying $\lambda^2 + \lambda\mu + \mu^2 \neq 0$ then $M_{\lambda,\mu} = kQ_8(z-1)(\lambda(s-1) + \mu(t-1))$ runs over pairwise nonisomorphic indecomposable kQ_8-modules of period 2. It follows from Proposition 6.4 (iv) that the almost vanishing morphisms starting at these modules determine elements in the graded center of the stable module category of kQ_8 in any odd degree, whence the result in the case $Q = Q_8$. For Q a generalised quaternion 2-group we may identify Q_8 to a subgroup of Q, and then the induction functor $\mathrm{Ind}_{Q_8}^Q$ sends the indecomposable modules of period 2 considered in Proposition 6.4 to indecomposable kQ-modules of period 2. Moreover, it follows from Proposition 6.4 (iv) in conjunction with Lemma 2.4 that the almost vanishing morphisms starting at these modules are again invariant under Ω_{kQ}^2, hence define elements in the graded center of the stable category of kQ-modules. $\qquad\square$

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